NEW EXAMPLES OF TORSION-FREE NON-UNIQUE PRODUCT GROUPS

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Abstract. We give an infinite family of torsion-free groups that do not satisfy the unique product property. For these examples, we also show that each group contains arbitrarily large sets whose square has no uniquely represented element.

1. Introduction

If $k[G]$ is a group ring over a torsion-free group, two natural questions that can be asked are what are the zero divisors, and what are the units? Both questions are very well known and considered to be two of the least tractable questions in the theory of group rings. A detailed discussion of the history of these problems (and other interesting open questions) can be found in [6].

Conjecture 1. Zero Divisor Conjecture (Kaplansky) *If G is a torsion-free group and K is an integral domain, then the group ring $K[G]$ has no zero divisors.*

Similarly, the second conjecture, which implies Conjecture 1, can be stated as.

Conjecture 2. Nontrivial Units Conjecture (Kaplansky) *If G is a torsion-free group and K is a field, then the only units in $K[G]$ are the trivial ones, i.e. those of the form kg where $k \in K$ and $g \in G$.*

The unique product property was initially conceived as an attempt to solve these conjectures. A group is G is said to satisfy the *unique product property* if given any two non-empty finite sets X, $Y \subset G$ then at least one element, say z in the product set $XY = \{xy \mid x \in X \text{ and } y \in Y\}$ can be written uniquely as a product, $z = xy$ where $x \in X$ and $y \in Y$. Many familiar groups satisfy this property, for example, orderable groups [6], diffuse groups [1] and locally indicable groups [2]. In particular, it is well known that every right orderable group satisfies this property. The converse, however, is still open.

Any group with torsion does not satisfy the unique product property, so the only interesting examples of groups without this property would necessarily be torsion-free. There are only two known examples of torsion-free groups that do not satisfy the unique product property (excluding, of course, torsion-free groups that contain either of these two examples as a proper subgroup).

The first example was given by E. Rips and and Y. Segev. The authors showed that there exists a family of torsion-free groups that do not satisfy this property [8]. In their examples, given predetermined sets, relations for a group were carefully constructed that in such a way that the resulting group is torsion-free and contains the two sets as a pair of non-unique product sets. Many seemingly natural questions regarding these groups are still open. In particular, nothing is known about these groups in relation to Conjectures 1 or 2.

The second known example of a group that does not satisfy the unique product property and the only known explicit example of such a group was given by D. Promislow in [7]. By means of a random search algorithm, he found a 14 element set S in the group

$$P = \langle x, y \mid xy^2x^{-1}y^2,\ yx^2y^{-1}x^2\rangle$$

with the property that SS has no uniquely represented element. We will call such a set S a *non-unique product set*. Given the nature of the search, very little is known about other non-unique product sets in P or about how to extend this result to other groups.

A result due to Lewin, [5], shows that P satisfies Conjecture 1.

Theorem 1. (Lewin) *If $G = G_1 *_{G_N} G_2$ a free product with amalgamation, where*

(1) *G_N is normal in both G_1 and G_2;*
(2) *$F[G_1]$ and $F[G_2]$ have no zero divisors;*
(3) *$F[G_N]$ satisfies the Ore condition.*

Then $F[G]$ has no zero divisors.

To see this, note that $P \cong K *_{\mathbb{Z}^2} K$, where K is a Klein bottle group and we identify index 2 subgroups that are isomorphic to $\mathbb{Z}^2$ in each copy of K. The second condition holds since group rings over locally indicable groups satisfy Conjecture 1. For the last condition, it is well known that a group ring over an abelian group satisfies the Ore condition. It is still unknown whether P satisfies Conjecture 2.

The purpose of this paper is to generate new simple examples of groups that do not satisfy the unique product property and to produce non-unique product sets whose existence can be inferred from the relations in the group. Currently, it is not all together clear where to look for such groups or even sets within these groups. Certainly these groups must be non-left orderable. In fact, this is precisely why P was initially seen as a likely candidate [4]; however, this does not tell us how to find such sets or even if they exist (clearly, any finite pair of subsets will not work). The hope is that generating more examples will lead to a better understanding of the structure of such groups. In Section 4, we do so by generalizing P in the following way.

Theorem 2. *For each $k > 0$, the torsion-free group*

$$P_k = \langle a, b, \mid ab^{2^k}a^{-1}b^{2^k},\ ba^2b^{-1}a^2\rangle$$

does not satisfy the unique product property, and for $k > 1$, does not contain P.

Note that the group P_1 is the same as Promisow's example P. The relations of P_1 and P_k are similar, but the groups are quite different. For example, it is well known that P is a finite extension of $\mathbb{Z}^3$ and as such is supersolvable. In contrast, the groups P_k for $k > 1$ are much larger. One can show P_k contains a finite index subgroup isomorphic to $\mathbb{Z}^2 \times F$, where F is a finitely generated free group. In particular, these groups are also not amenable and hence are not solvable. An argument, identical to the one above, shows that each P_k satisfies the hypotheses of Theorem 1 and thus every group P_k satisfies Conjecture 1.

These groups are generalizations of P in the sense that each P_k is is an amalgamation of Klein bottle groups over $\mathbb{Z}^2$. However, we wish to emphasize that the non-unique product sets we construct in Section 4 are not generalizations of Promislow's set S found [7], but rather arise from a careful study of the geometry of

the Cayley graph given by the presentation above. Roughly, the idea is to construct specific paths in the Cayley graph taken sufficiently long so that the Klein bottle relations force certain paths from the product set to overlap nicely. In Section 5, this idea is extended to longer paths in the Cayley graph to prove the following result.

Theorem 3. *Each group P_k contains arbitrarily large non-unique product sets.*

2. Preliminaries

If a group G acts by automorphisms on a simplicial tree T, without inversion, then T is called a G-tree. The action is said to be *trivial* if G fixes a point and *minimal* if there is no invariant G-subtree.

In this setting, an automorphism is said to be *elliptic* if it fixes a point and *hyperbolic* otherwise. If g is elliptic, we define $Fix(g)$ to be the set of all points fixed by g. Following [9], we can characterize these automorphisms in the following way.

Proposition 1. *Let G be group that acts on a simplicial tree T by automorphisms.*

(1) *If $g \in G$, then either g acts on a unique simplicial line in T by translations or $Fix(g) \neq \emptyset$.*
(2) *If $g_1,\ g_2 \in G$ and $Fix(g_1),\ Fix(g_2)$ are nonempty and disjoint, then $Fix(g_1 g_2) = \emptyset$.*
(3) *If G is generated by a finite set of elements $s_1,\ s_2,\ \ldots,\ s_m$ such that s_j and $s_i s_j$ fix points in T for all $i,\ j$, then G the action of G is trivial.*

The unique simplicial line in (1) is called the *axis* of g and denoted A_g. Further, following [3], we can describe minimal subtrees in the following way.

Proposition 2. *If G is finitely generated and T is a non-trivial G-tree then T contains a unique minimal G-invariant subtree, which is the union of the axes of all the hyperbolic elements in G.*

A natural setting for groups acting on G-trees is when G splits as a free product with amalgamation, an HNN extension, or more generally as the fundamental group of a graph of groups. From [9] there exists a tree T, referred to as the Bass-Serre tree, on which G acts simplicially. For our purposes, we need only consider the case in which $G \cong A *_C B$. In this case, such a tree is described as follows. The vertices of the tree T are given by $G/A \cup G/B$. The edges are given by G/C, with initial vertices $v_i(gC) = gA$ and the terminal vertices $v_t(gC) = gB$. The stabilizers of the vertices are the conjugates of A and B, and the edge stabilizers are the conjugates of C.

3. Properties of the Groups P_k

Note that just as in P, each group P_k is a free product with amalgamation. To see this, fix $k > 0$, and take two Klein bottle groups

$$K_1 = \langle a,\ x \mid axa^{-1}x\rangle \text{ and } K_2 = \langle y,\ b \mid byb^{-1}y\rangle$$

with subgroups

$$A_1 = \langle a^2,\ x\rangle \cong \mathbb{Z}^2 \text{ and } A_2 = \langle b^{2^k},\ y\rangle \cong \mathbb{Z}^2.$$

respectively. If we define the isomorphism

$$\phi : A_1 \to A_2 \text{ by } x \mapsto b^{2^k} \text{ and } a^2 \mapsto y,$$

then the free product of K_1 and K_2 with amalgamation of A_1 and A_2, by ϕ has the presentation

$$K_1 *_{A_1} K_2 \cong \langle a,\ b,\ x,\ y \mid axa^{-1}x,\ byb^{-1}y,\ x = b^{2^k} y = a^2 \rangle \cong P_k.$$

For concreteness, we will choose transversal

$$T_{K_1} = \{1,\ a\} \text{ and } T_{K_2} = \{1,\ b,\ \ldots,\ b^{2^k-1}\}.$$

So, as an amalgamated product with transversal T_{K_1} we have the following results.

Proposition 3. (Normal Forms)

Every element $w \in P_k$ can be written uniquely in the form:

$$w = a^{2u} b^{2^k v} a^{\alpha} b^{\beta_1} a b^{\beta_2} a \ldots b^{\beta_l} a b^{\beta}$$

where $u,\ v \in \mathbb{Z}$, $\alpha \in \{0,\ 1\}$, $\beta_i \in \{1,\ b,\ \ldots,\ b^{2^k-1}\}$, and $\beta \in \{0,\ 1,\ b,\ \ldots,\ b^{2^k-1}\}$

As an amalgamated product of torsion-free groups, from [9] we have

Proposition 4. *Every group P_k is torsion-free.*

Ultimately, we want to show that every group P_k does not satisfy the unique product property and hence gives an infinite family of simple concrete examples. One issue that needs to be addressed is that some of the groups P_k $(k > 1)$ could contain P and hence not be truly new examples. We will show that every group does not contain P. This will be done by showing the following:

- If $A,\ B \in P_k$ where $\langle A,\ B\rangle$ fixes a line L in P_k, and $\langle A,\ B\rangle$ acts on L with no global fixed point, then the relations

$$AB^2A^{-1}B^2 = 1 \text{ and } BA^2B^{-1}A^2 = 1$$

  can not simultaneously hold in P_k.
- If $P \leq P_k$, then the induced action of P on P_k fixes a line L_k in T_k.

Lemma 1. *Suppose $\langle A,\ B\rangle$ fixes a line L in T_k. If A and B are hyperbolic, then neither of the relations*

$$AB^2A^{-1}B^2 = 1 \textit{ and } BA^2B^{-1}A^2 = 1$$

can hold in P_k.

Proof. Suppose A and B are hyperbolic elements that stabilize the same line U. Then there $m,\ n \in \mathbb{Z}$ so that A^nB^{-m} fixes L pointwise. So $A^nB^{-m} \in \langle a^2,\ b^{2^k}\rangle$ or rather $A^n = a^{2s}b^{2^k t}B^m$, for some $s,\ t \in \mathbb{Z}$. So if the relation $BA^2B^{-1}A^2 = 1$ holds, then so does $1 = BA^{2n}B^{-1}A^{2n}$. It follows then that $B^{4m} \in \langle a^2,\ b^{2^k}\rangle$, contradicting the fact that B is hyperbolic. A similar result holds if we assume that $AB^2A^{-1}B^2 = 1$ holds. □

Lemma 2. *If A is hyperbolic and B is elliptic, then the following relations*

$$AB^2A^{-1}B^2 = 1 \textit{ and } BA^2B^{-1}A^2 = 1$$

can not simultaneously hold in P_k.

Proof. Suppose otherwise. Conjugating if necessary, we may assume that either

$$B = a^{2s}b^{2^k t}a \text{ or } B = a^{2s}b^{2^k t}b^{2^{k-1}}$$

and from Proposition 3 we may write

$$A = a^{2u}b^{2^k v}a^{\alpha}b^{\beta_1}ab^{\beta_2}a\dots b^{\beta_1}ab^{\beta}$$

as a word in reduced normal form. In either case of B, the idea of the proof is to analyze the possible values of $\alpha,\ \beta_1,\ \beta_2,\ \dots,\ \beta_1,$ and β, and show that no such word A exists.

Consider the case $B = a^{2s}b^{2^k}a$. The first relation says that

$$1 = AB^2A^{-1}B^2 = a^{4s+2+\sigma_b(A)(4s+2)}$$

which is true if and only if $\sigma_b(A) = -1$, where

$$\sigma_b(A) = \begin{cases} 1 & \text{if the sum of all the powers of } b \text{ in } A \text{ is even} \\ -1 & \text{if the sum of all the powers of } b \text{ in } A \text{ odd} \end{cases}.$$

Suppose the relation

$$(1) \qquad 1 = BA^2B^{-1}A^2 = a^{2q}b^{2^k r}a(a^{\alpha}b^{\beta_1}a\dots b^{\beta_l}ab^{\beta})^2a^{-1}(a^{\alpha}b^{\beta_1}a\dots b^{\beta_l}ab^{\beta})^2.$$

holds. By assumption, A is a hyperbolic element, and so $A^2 \notin \langle a^2,\ b^{2^k}\rangle$. We claim that cancellation must occur in the subword $ab^{\beta}a^{-1}a^{\alpha}b^{\beta_1}$. Otherwise, say in the case where $\alpha = 0$ and $\beta \neq 0$, then the right hand side of (1) above can be written as a non-trivial word in normal form contradicting Proposition 3. Similarly, in the case where $\alpha = 1$ and $\beta = 0$, the right hand side of (1) reduces to a non-trivial word in normal form, which also contradicts Proposition 3. Hence, the only cases that need to be considered are when $\alpha = 0$ and $\beta = 0$ or when $\alpha = 1$ and $\beta \neq 0$. We will handle both cases at the same time, so for concreteness, relabel $\beta = \beta_{l+1}$. After reduction of the pair aa^{-1}, right hand side of (1) contains a subword of the form $b^{\beta_i+\beta_j}$. If $\beta_i + \beta_j = 2^k$, move $b^{\beta_i+\beta_j}$ and the resulting a^2 to the far left in (1) as described by Proposition 3. Repeat this process for the next resulting subword $b^{\beta_{i-1}+\beta_{j+1}}$. If at any stage of the reduction, we have $b^{\beta_s+\beta_t} \neq 2^k$, then the reduced word in (1) is a non-trivial word in normal form, leading to a contradiction of Proposition 3. Pairing off the powers of b in this way, we have either:

(1) $\alpha = 0,\ \beta = 0,\ \beta_l + \beta_1 = 2^k,\ \beta_{l-1} + \beta_2 = 2^k,\ \dots,\ \beta_{\frac{l}{2}+1} + \beta_{\frac{l}{2}} = 2^k$ (if l is even),
(2) $\alpha = 0,\ \beta = 0,\ \beta_l + \beta_1 = 2^k,\ \beta_{l-1} + \beta_2 = 2^k,\ \dots,\ \beta_{\frac{l+1}{2}} + \beta_{\frac{l+1}{2}} = 2^k$ (if l is odd),
(3) $\alpha = 1,\ \beta \neq 0,\ \beta + \beta_1 = 2^k,\ \beta_l + \beta_2 = 2^k,\ \dots,\ \beta_{\frac{l+2}{2}} + \beta_{\frac{l+2}{2}} = 2^k$ (if l is even), or
(4) $\alpha = 1,\ \beta \neq 0,\ \beta + \beta_1 = 2^k,\ \beta_l + \beta_2 = 2^k,\ \dots,\ \beta_{\frac{l+1}{2}+1} + \beta_{\frac{l+1}{2}} = 2^k$ (if l is odd),

In any event, this forces $\sigma_b(A) = 1$ giving a contradiction.

Consider the other case, where $B = a^{2s}b^{2^k t}b^{2^{k-1}}$. Using the same normal form for A as above, the relation

$$1 = AB^2A^{-1}B^2 = a^{4s+\sigma_b(A)4s}b^{2^{k+1}t+2^k+\sigma_a(A)(2^{k+1}t+2^k)}$$

holds provided $\sigma_a(A) = -1$ and either $\sigma_b(A) = -1$ or $s = 0$, where

$$\sigma_a(w) = \begin{cases} 1 & \text{if the sum of all the powers of } a \text{ in } w \text{ is even} \\ -1 & \text{if the sum of all the powers of } a \text{ in } w \text{ odd} \end{cases}$$

and $\sigma_b(A)$ is as above.

An argument similar to the one above applied to the relation

$$1 = BA^2B^{-1}A^2$$

shows

(1) $\alpha = 0$, $\beta \neq 0$, $\beta + \beta_1 = 2^{k-1}$, $\beta_l + \beta_2 = 2^k$, …, $\beta_{\frac{l+1}{2}+1} + \beta_{\frac{l+1}{2}} = 2^k$,
(2) $\alpha = 0$, $\beta = 0$, $\beta_1 = 2^{k-1}$, $\beta_l + \beta_2 = 2^k$, …, $\beta_{\frac{l+1}{2}} + \beta_{\frac{l+1}{2}+1} = 2^k$, or
(3) $\alpha = 1$, $\beta = 2^{k-1}$, $\beta_l + \beta_1 = 2^k$, $\beta_{l-1} + \beta_2 = 2^k$, …, $\beta_{\frac{l}{2}} + \beta_{\frac{l}{2}+1} = 2^k$.

and so in every case, $\sigma_b(A) = 1$.

So we must have that $s = 0$. If we simply count the number of exponents in a of $BA^2B^{-1}A^2$, one checks that after all possible cancellations, this is $8u + 4(2j + 1)$ for some integer j, i.e. this is true by our description of A and B if no cancellations occur and any cancellation reduces the total number of exponents in a by 8. Since $8u + 4(2j + 1) = 0$ has no integer solution, this relation holding would contradict Proposition 4. □

Lemma 3. *If $\langle A,\ B\rangle \subset P_k$ fixes some line L in the Bass-Serre Tree T_k where A and B are elliptic elements with disjoint fixed point sets, then the following relations*

$$AB^2A^{-1}B^2 = 1 \text{ and } BA^2B^{-1}A^2 = 1$$

can not simultaneously hold in P_k.

Proof. If A and B are elliptic elements with disjoint fixed point sets, then AB acts as a translation on U. Moreover that $\langle AB,\ B\rangle = \langle AB,\ B\rangle$ and if A and B satisfy the relations above, then so do AB and B. So $\langle AB,\ B\rangle$ satisfied the hypotheses of the preceding lemma and both relations which contradicts the preceding lemma. □

Theorem 4. *For $k > 1$, P_k does not contain P.*

Proof. Fix $k > 1$ and suppose that $\langle A,\ B\rangle \cong P$ is a subgroup of P_k. Since P_k acts on the Bass-Serre tree T_k, there is an induced action of P on T_k by isometries without edge inversion. It follows that the action of P on T_k has no global fixed point; otherwise, $P \leq K_1^g$ or $P \leq K_2^g$ for some $g \in P_k$ and in particular, this implies that the surface groups K_1^g or K_2^g contain a free Abelian group of rank 3. Since P is finitely generated and T_k is non-trivial, by Proposition 2, T_k contains a unique minimal P-invariant subtree which we will denote by L. By Proposition 1, L contains at least one axis. On the other hand, since P is a finite extension of $\mathbb{Z}^3$, the largest tree P can act on is a line. So, if P is a subgroup of P_k, we can deduce that P acts simplicially on a line $L \subset T_k$. Applying Lemmas 1, 2, and 3 gives us the desired contradiction. □

4. Family of groups

Let k be a fixed positive integer that we will use for the remainder of the paper. In this section, we will show that P_k does not satisfy the unique product property. Recall, that given a torsion-free group G, a subset of the form $\{xr^i \mid l \leq i \leq m\}$

for some x, $r \in G$ and $l, m \in \mathbb{Z}$ is said to be a *left progression of ratio* r, or simply a *left* r*-progression.* In P_k, consider the following b-progressions

$$X_0 = \{a^{-1},\ a^{-1}b\},$$
$$X_i = \{b^i a^{-1} b^j \mid\ 0 \leq j \leq 2^k + 1\},$$
$$Y_l = \{b^l a b^j \mid\ 1 \leq j \leq 2^k + 1\},$$
$$Z_0 = \{b^j \mid\ -2^k \leq j \leq 2^k\}$$

where $1 \leq i \leq 2^k - 1$ and $0 \leq l \leq 2^k - 1$. Set

$$T = \bigcup_{i=0}^{2^k-1} X_i \cup \bigcup_{j=0}^{2^k-1} Y_j \cup Z_0$$

and for convenience, set $\bigcup_{i=0}^{2^k-1} X_i$ and $Y = \bigcup_{j=0}^{2^k-1} Y_j$. Proposition 3 shows every element in T is distinct we will show that every element in TT has no unique representation as follows. First, decompose TT into smaller product sets of the form

$$X_iX_j,\ Y_iX_j,\ X_iY_j,\ Y_iY_j,\ Z_0X_i,\ X_iZ_0,\ Z_0Y_i,\ Y_iZ_0,\ \text{ and } Z_0Z_0.$$

From there, we decompose these product sets further into progressions that are obtained as the product of single element in T with one of the sets Y_j, X_i, or Z_0, which we will refer to as *slices*.

Showing TT is a non-unique product set requires careful bookkeeping to make keeping track of the specific slices easier, we will adopt the following conventions. Write $x_{(n,m)} = b^n a^{-1} b^m$, $y_{(n,m)} = b^n a b^m$, and $z_{(0,n)} = b^n$ and if $u_{(m,i)} \in T$ and $W_n = \{w_{(n,j)} \mid l_n \leq j \leq m_n\}$ is one of our b-progressions listed above, we will denote the slices by

$$u_{(m,i)}W_n = \{u_{(m,i)}w_{(n,j)} \mid l_n \leq j \leq m_n\}.$$

Clearly, any product in TT that belongs to two of these slices has two different representations in TT. Using the our choice of the b-progressions, we can efficiently show most of these slices are contained in at least one other slice. This reduces the number of elements we need to check to a much smaller set. For the remaining slices, the Klein bottle relations are used to show the remaining slices are contained in at least two of the subproduct sets listed above and hence have two distinct representations.

The following equalities and containments hold for subproduct sets in TT as a result of the structure of the progressions. These are perhaps easiest to see visually, as in figures 1, 2, and 3, by writing the respective products U_iY,U_iX, and U_iZ_0 in table form, where U_i is an arbitrary progression in T. In figures 1 and 2, the rows are labeled by individual words in a progression (written in order from the starting value $u_{i,s}$ to the ending value $u_{i,e}$) and the columns are labeled by the progressions in X and Y respectively. In figure 3, both row and column are labeled by words in the respective progressions (also written in the order of the progression). In each the figures, the circled slices are those that are not paired up by the structure of the progressions mentioned above.

Case 1: Consider products of the form U_iY. As illustrated in figure 1, the slices along the diagonal lines are equal since we always have

$$u_{(i,v+1)}Y_w = \{b^i a^\epsilon b^{v+1} b^w a b^j \mid 1 \leq j \leq 2^k + 1\} = u_{(i,v)}Y_{w+1},$$

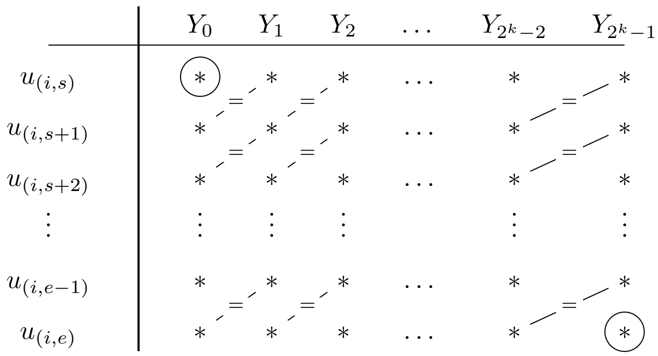

FIGURE 1. Matching Patterns for Products of the Form U_iY

where $\epsilon \in \{-1,\ 0,\ 1\}$ and w and v are taken in the appropriate range. So the only slices we need consider separately, are those of the form

$$u_{(i,s)}Y_0 \text{ and } u_{(i,e)}Y_{2^k-1}$$

for appropriate starting values s and ending values e of each progression.

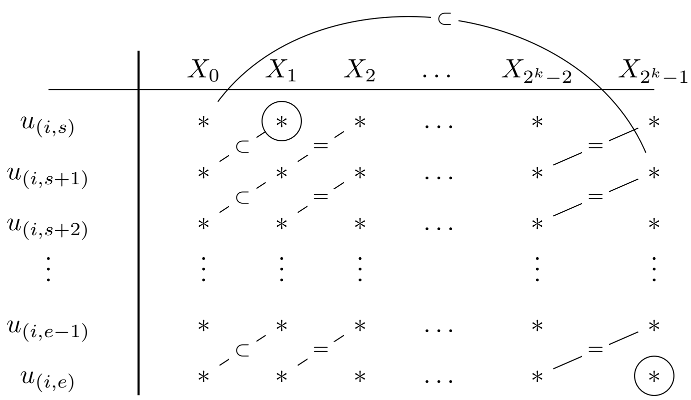

FIGURE 2. Matching Patterns for Products of the Form U_iX

Case 2 Consider products of the form U_iX. Just as in Case 1, we have similar identifications along the diagonal lines for all the slices with the same cardinality, as illustrated in figure 2. However, we also have proper containments, since the slices $u_{(i,j)}X_0$ only have cardinality 2. There are two containments of particular interest, namely $u_{(i,s)}X_0 \subset u_{(i,s+1)}X_{2^k-1}$ and $u_{(i,s+1)}X_0 \subset u_{(i,s)}X_1$. The former always occurs, since

$$u_{(i,s)}X_0 = \{b^ia^\epsilon b^sa^{-1}b^j \mid j = 0,\ 1\} \subset \{b^ia^\epsilon b^sa^{-1}b^j \mid -2^k \leq j \leq 1\} = u_{(i,s+1)}X_{2^k-1}.$$

Containment in the latter case is clear, but it is worth mentioning this containment plays a very important role, later. Therefore, the only slices we need to consider separately are those of the form

$$u_{(i,e)}X_{2^k-1} \text{ and the shortened } u_{(i,s)}X_1 \text{ written as } \{u_{(i,s)}ba^{-1}b^j \mid 2 \leq j \leq 2^k+1\},$$

where once again s and e are the appropriate starting and ending values of the progression U_i.

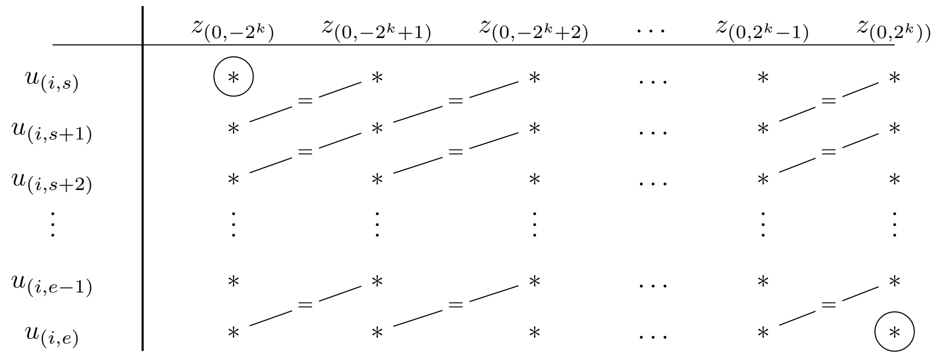

FIGURE 3. Matching Patterns for Products of the Form $U_i Z_0$

Case 3: Consider products of the form $U_i Z_0$. As illustrated in figure 3, each product has exactly two elements $\{u_{(i,s)}b^{-2^k},\ u_{(i,e)}b^{2^k}\}$ that are not identified within the table. If $U_i \neq Z_0$, then it is clear that

$$u_{(i,s)}b^{-2^k} = b^{2^k}u_{(i,s)} \subset Z_0U_i \text{ and } u_{(i,s)}b^{2^k} = b^{-2^k}u_{(i,s)} \subset Z_0U_i$$

and Z_0U_i is contained in either Z_0X or Z_0Y. Hence, these elements have no unique representation in TT. If $U_i = Z_0$, the elements not identified within the table are $\{b^{-2^{k+1}},\ b^{2^{k+1}}\}$. Since we have

$$b^{-2^{k+1}} = ab^{2^k+1}b^{2^k-1}a^{-1} \in y_{(0,2^k+1)}X_{2^k-1}$$

$$b^{2^{k+1}} = b^{2^k-1}a^{-1}ab^{2^k+1} \in x_{(2^k-1,0)}Y_0.$$

these elements also have no unique representation in TT.

We can extend this idea further to account for the remaining slices in Z_0Y and Z_0X. As illustrated in figure 1, the slices we have yet to account for in the subproduct set Z_0Y are

$$z_{(0,-2^k)}Y_0 = \{ab^j \mid 2^k + 1 \leq j \leq 2^{k+1} + 1\} \subset Y_0Z_0$$

and

$$z_{(0,2^k)}Y_{2^k-1} = \{b^{2^k-1}ab^j \mid 1 - 2^k \leq j \leq 1\} \subset Y_{2^k-1}Z_0.$$

Similarly, as illustrated in figure 2, the slices we have yet to account for in the subproduct set Z_0X are subsets of the slices

$$z_{(0,-2^k)}X_1 = \{ba^{-1}b^j \mid 2^k \leq 2^{k+1} + 1\} \subset X_1Z_0$$

and

$$z_{(0,2^k)}X_{2^k-1} = \{b^{2^k-1}a^{-1}b^j \mid -2^k \leq j \leq 1\} \subset X_{2^k-1}Z_0.$$

This accounts for all the subproduct sets of the form Z_0U_i and U_iZ_0.

Remaining Elements in TT	
Slice	Rewritten Elements
Remaining Values for j	
$x_{(0,0)}X_1$	$abab^j \subset Y_0Y_0$
$2 \le j \le 2^k + 1$	
$x_{(0,1)}X_{2^k-1}$	$a^{-2}b^j \subset Y_{2^k-1}Y_{2^k-1}$
$-2^k \le j \le 1$	
$x_{(l,0)}X_1$	$b^l abab^j \subset Y_lY_0$
$2 \le j \le 2^k + 1$	
$x_{(l,2^k+1)}X_{2^k-1}$	$a^2b^j \subset Y_{l-1}Y_{2^k-1}$
$l - 2^{k+1} \le j \le l + 1 - 2^k$	
$x_{(m,0)}X_1$	$b^m abab^j \subset Y_mY_0$
$2 \le j \le 2^k + 1$	
$x_{(m,2^k+1)}X_{2^k-1}$	$a^{-2}b^j \subset Y_{m-1}Y_{2^k-1}$
$m - 2^{k+1} \le j \le m + 1 - 2^k$	
$x_{(2^k-1,0)}X_1$	$b^{2^k-1}abab^j \subset Y_{2^k-1}Y_0$
$2 \le j \le 2^k + 1$	
$x_{(2^k-1,2^k+1)}X_{2^k-1}$	$a^2b^j \subset Y_{2^k-2}Y_{2^k-1}$
$-1 - 2^k \le j \le 0$	
$y_{(n,1)}X_1$	$b^na^{-1}b^2ab^j \subset X_nY_1$
$2 \le j \le 2^k + 1$	
$y_{(n,2^k+1)}X_{2^k-1}$	$b^j \subset Z_0Z_0$
$n - 2^{k+1} \le j \le n + 1 - 2^k$	
$y_{(0,1)}Y_0$	$abab^j \subset X_0X_1$
$1 \le j \le 2^k + 1$	
$y_{(0,2^k+1)}Y_{2^k-1}$	$a^2b^j \subset X_1X_{2^k-1}$
$1 - 2^{k+1} \le j \le 1 - 2^k$	
$y_{(l,1)}Y_0$	$b^l abab^j \subset X_lX_1$
$1 \le j \le 2^k + 1$	
$y_{(l,2^k+1)}Y_{2^k-1}$	$a^{-2}b^j \subset X_{l+1}X_{2^k-1}$
$l + 1 - 2^{k+1} \le j \le l + 1 - 2^k$	
$y_{(m,1)}Y_0$	$b^m abab^j \subset X_mX_1$
$1 \le j \le 2^k + 1$	
$y_{(m,2^k+1)}Y_{2^k-1}$	$a^2b^j \subset X_{m+1}X_{2^k-1}$
$m + 1 - 2^{k+1} \le j \le m + 1 - 2^k$	
$y_{(2^k-1,1)}Y_0$	$b^{2^k-1}abab^j \subset X_{2^k-1}X_1$
$1 \le j \le 2^k + 1$	
$y_{(2^k-1,2^k+1)}Y_{2^k-1}$	$a^{-2}b^j \subset X_0X_{2^k-1}$
$-2^k \le j \le 0$	
$x_{(n,0)}Y_0$	$b^j \subset Z_0Z_0$
$n + 1 \le j \le n + 1 + 2^k$	
$x_{(0,1)}Y_{2^k-1}$	$b^j \subset Z_0Z_0$
$1 - 2^k \le j \le 1$	
$x_{(n,2^k+1)}Y_{2^k-1}$	$b^j \subset Z_0Z_0$
$n + 1 - 2^{k+1} \le j \le n + 1 - 2^k$	

As mentioned above, we will show that these sets are contained in two of the smaller product sets. Clearly, $u_{(m,i)}W_n \subset U_m W_n$, so for each remaining slice, we need only find some other product set that contains it. In the chart given above, we list all of the remaining slices as well as a reduced form for each of the words obtained by applying the relations

$$aba = a^{-1}ba^{-1},\ ba^2b^{-1}a^2 = 1, \text{ and } ab^{2^k}a^{-1}b^{-2^k} = 1,$$

where

$$l \in \{1,\ 3,\ 5,\ \ldots,\ 2^k{-}3\},\ m \in \{2,\ 4,\ 6,\ \ldots,\ 2^k{-}2\}, \text{ and } n \in \{0,\ 1,\ 2,\ \ldots,\ 2^k{-}1\}.$$

In case 2 above, we used a smaller progression X_0 to shorten the length of the remaining words in the slices $u_{(i,s)}X_1$ so that they will fit inside the product sets Y_iY_0. In the chart above, we will list only those elements that have not been accounted for by the structure of the progressions. In each case, containment is verified by considering the reduced words and length of the remaining values in j. Inspection shows that every product in TT is not uniquely represented, and so T is a non-unique product set. Since k is arbitrary, this shows that each P_k does not have the unique product property.

5. Cardinalities of Non-unique Product Sets

From the standpoint of Conjectures 1 and 2, it seems natural to consider the cardinality of the possible non-unique product sets in G. Indeed if the cardinality of such sets were bounded, then one need only consider products in $k[G]$ of bounded support size. In this section, we will show that this is not possible in general, by showing that each P_k contains arbitrarily large square non-unique product sets.

The construction in the preceding section shows that P_k contains T a set with cardinality $2^{2k+1}+2^{k+2}+1$ with the property that TT has no uniquely represented elements. We will construct larger sets as follows. Let p be any fixed positive odd integer and choose an odd integer q so that $q-1$ is a multiple of 2^k. For these odd integers p and q, consider the following b-progressions in P_k.

$$\begin{aligned} X_0(p,q) &= \{a^{-p}b^j \ \mid\ -q+1 \le j \le (2^k+1)q - 2^k\}, \\ X_i(p,q) &= \{b^i a^{-p} b^j \ \mid\ -q+1 \le j \le (2^k+1)q\}, \\ Y_l(p,q) &= \{b^l a^p b^j \ \mid\ -q+2 \le j \le (2^k+1)q\}, \\ Z_0(p,q) &= \{b^j \ \mid\ -2^k(\frac{q+1}{2}) - (q-1) \le j \le 2^k(\frac{q+1}{2}) + (q-1)\} \end{aligned}$$

where $0 \le i \le 2^k-1$ and $0 \le j \le 2^k-1$. We want to show that

$$T(p,q) = \bigcup_{i=0}^{2^k-1} X_i(p,q) \cup \bigcup_{j=0}^{2^k-1} Y_j(p,q) \cup Z_0(p,q) \subset P_k$$

has the property that the product set $T(p,q)T(p,q)$ has no uniquely represented element.

Once again, our normal forms ensure that every element in $T(p,q)$ is distinct. The method of showing this set has no uniquely represented element is analogous to the case where p and q are 1, as given in Section 4. In fact, the matchings in figures 1, 2, and 3 are identical here as well. Given this similarity, we will only list those elements that are not matched via the progressions in the table below.

Remaining Elements in $T(p,q)T(p,q)$	
Slice	Rewritten Elements
Remaining Values for j	
$x_{(0,-q+1)}X_1$	$a^pba^pb^j \subset Y_0Y_0$
$2^kq+2q-2^k \leq j \leq 2^kq+2q-1$	
$x_{(0,(2^k+1)q-2^k)}X_{2^k-1}$	$a^{-2p}b^j \subset Y_{2^k-1}Y_{2^k-1)}$
$2-2^kq-2q \leq j \leq 1$	
$x_{(l,-q+1)}X_1$	$b^la^pba^pb^j \subset Y_lY_0$
$2^kq+2q-2^k \leq j \leq 2^kq+2q-1$	
$x_{(l,(2^k+1)q)}X_{2^k-1}$	$a^{2p}b^j \subset Y_{l-1}Y_{2^k-1}$
$l+2-2^kq-2q-2^k \leq j \leq l+1-2^k$	
$x_{(m,-q+1)}X_1$	$b^ma^{-p}ba^{-p}b^j \subset Y_mY_0$
$2^kq+2q-2^k \leq j \leq 2^k+2q-1$	
$x_{(m,(2^k+1)q)}X_{2^k-1}$	$a^{-2p}b^j \subset Y_{m-1}Y_{2^k-1}$
$m+2-2^kq-2q-2^k \leq j \leq m+1-2^k$	
$x_{(2^k-1,-q+1)}X_1$	$b^{2^k-1}a^pba^pb^j \subset Y_{2^k-1}Y_0$
$2^kq+2q-2^k \leq j \leq 2^kq+2q-1$	
$x_{(2^k-1,(2^k+1)q)}X_{2^k-1}$	$a^{2p}b^j \subset Y_{2^k-2}Y_{2^k-1}$
$1-2^kq-2q \leq j \leq 0$	
$y_{(n,-q+2)}X_1$	$b^na^pb^2a^{-p}b^j \subset X_nY_1$
$2^kq+2q-2^k \leq j \leq 2^kq+2q-1$	
$y_{(n,(2^k+1)q)}X_{2^k-1}$	$b^j \subset Z_0Z_0$
$n+2-2^kq-2q-2^k \leq j \leq n+1-2^k$	
$y_{(0,-q+2)}Y_0$	$a^pba^pb^j \subset X_0X_1$
$1 \leq j \leq 2^kq+2q-1$	
$y_{(0,(2^k+1)q)}Y_{2^k-1}$	$a^{2p}b^j \subset X_1X_{2^k-1}$
$3-2^kq-2q-2^k \leq j \leq 1-2^k$	
$y_{(l,-q+2)}Y_0$	$b^la^pba^pb^j subset X_lX_1$
$1 \leq j \leq 2^kq+2q-1$	
$y_{(l,(2^k+1)q)}Y_{2^k-1}$	$a^{-2p}b^j \subset X_{l+1}X_{2^k-1}$
$l+3-2^kq-2q-2^k \leq j \leq l+1-2^k$	
$y_{(m,-q+2)}Y_0$	$b^ma^pba^pb^j \subset X_mX_1$
$1 \leq j \leq 2^kq+2q-1$	
$y_{(m,(2^k+1)q)}Y_{2^k-1}$	$a^{2p}b^j \subset X_{m+1}X_{2^k-1}$
$m+3-2^kq-2q-2^k \leq j \leq m+1-2^k$	
$y_{(2^k-1,-q+2)}Y_0$	$b^{2^k-1}a^pba^pb^j \subset X_{2^k-1}X_1$
$1 \leq j \leq 2^q+2q-1$	
$y_{(2^k-1,(2^k+1)q)}Y_{2^k-1}$	$a^{-2p}b^j \subset X_0X_{2^k-1}$
$2-2^kq-2q \leq j \leq 0$	
$x_{(n,-q+1)}Y_0$	$b^j \subset Z_0Z_0$
$n+1 \leq j \leq n+2^kq+2q-1$	
$x_{(0,(2^k+1)q-2^k)}Y_{2^k-1}$	$b^j \subset Z_0Z_0$
$3-2^kq-2q \leq j \leq 1$	
$x_{(n,(2^k+1)q)}Y_{2^k-1}$	$b^j \subset Z_0Z_0$
$n+3-2^kq-2q-2^k \leq j \leq n+1-2^k$	

For the remaining slices, we summarize the results in the table above (suppressing (p, q)), the argument is similar to results in Section 4 and the words the slices are rewritten using the relations

$$a^p b a^p = a^{-p} b a^{-p},\ b a^{2p} b^{-1} a^{2p} = 1, \text{ and } a^p b^{2^k} a^{-p} b^{2^k} = 1,$$

in P_k. As a result of the containments, $T(p, q)$ is also a non-unique product set. Further, note that our construction does not depend on a specific choice of p and q. Since each set $T(p, q) \subset P_k$ has cardinality $(2^{2k+1} + 5 \times 2^k + 2)q - (2^k + 1)$ this establishes Theorem 3. In our construction, we only needed that p was an odd positive integer, if we consider

$$\{T(2n-1,\ q) \mid\ n \geq 1 \text{ and } q-1 \text{ is a fixed multiple of } 2^k\},$$

this also shows there are infinitely many distinct square non-unique product sets for any fixed cardinality.

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